On the Lateral Vibrations of Bars of Conical Type.

By Dorothy Wrinch, D.Sc., Girton College, Cambridge.

(Communicated by Prof. Arthur Dendy, F.R.S. Received April 4, 1922.)

Introduction.

This paper contains a discussion of the lateral vibrations of a thin conical bar of circular section, which has its tip free. By means of a discussion of the roots of certain equations containing Bessel functions of the second and third orders, both of real and of imaginary argument, the frequencies and nodal arrangement associated with the first three tones are investigated, in the case when the base of the bar is clamped.

The lateral vibrations of conical bars of circular section were first treated by Kirchhoff,* in 1879. In his investigations, Kirchhoff was concerned with the case of a bar with its tip free and its base clamped, and he limited himself to a discussion of the period associated with the gravest tone, and considered neither the higher periods nor the positions of the nodes associated with them. J. W. Nicholson,† in the course of an investigation of the lateral vibrations of certain types of bars of variable section, discussed the case of a double cone (consisting of two equal cones placed base to base,) vibrating with both tips free, and discussed the periods and nodal ratios associated with the first three symmetrical tones. His results, however, throw no light on the question of the vibrations of a conical bar with a clamped base, owing to the peculiar nature of the conditions at the centre.

The present paper, in addition, contains an investigation of the periods of the higher tones, and of the arrangement of the nodes associated with these tones. The discussion gives for the case of a clamped-free bar of conical type the results which correspond to those given by Lord Rayleigh,‡ and Seebeck,§ and Donkin, for a bar of uniform cross section. General characteristics are worked out of the nodal arrangement in the higher tones when the tip is free, and nothing is known about the conditions at the base.

In the discussion of the higher tones of the free-clamped bar, the method is used of determining the equation which gives the period by a generalised hypergeometric series of the form

---

* 'Sitz. der Akad. zu Berlin,' p. 815 (1879).

† 'Proc. Roy. Soc.,' A, p. 506 (1917).

‡ 'Theory of Sound,' vol. 1, p. 286.

§ 'Abhandlungen d. Math. Phys. Classe d. k. Sächs. Gesellschaft d. Wissenschaften, Leipzig, 1852.

$$f(a, b, c;\ x) = 1 - x/1\,.\,1+a\,.\,1+b\,.\,1+c$$
$$+x^2/1\,.\,2\,.\,1+a\,.\,2+a\,.\,1+b\,.\,2+b\,.\,1+c\,.\,2+c - \ldots,$$

and introducing the asymptotic expansion* of this series for large values of x, which has not hitherto been employed in physical problems. The first term alone of this expansion, which is a multiple of

$$\cos\left(4x^{1/4}\sin\pi/4 - \pi\,(a+b+c+3/2)/4\right),$$

when equated to zero, gives the periods with an accuracy of about one part in 100 in the case of the third tone, and with progressively greater accuracy for the higher tones.

The investigation yields the following result:—

In the lateral vibrations of a conical bar vibrating with a free end, whatever the conditions at the base may be, the nodes in the nth tone occur at intervals of order $1/n$ in the neighbourhood of the base, and at intervals of order $1/n^2$ in the neighbourhood of the tip.

The importance of vibrations in morphology has been pointed out by Prof. Dendy,† in the course of an investigation into the positions of siliceous deposits found on sponge spicules; and, indeed, the investigation of Prof. Nicholson, already referred to, was undertaken with the purpose of testing the suggestion of Prof. Dendy, that these deposits occur at the nodes of the spicules, regarded as vibrating rods. The high degree of agreement obtained gives this suggestion a large measure of probability. In the types of physical problems which have been discussed from this point of view, the lower notes have, for the most part, been the important ones. With further applications to problems of morphology in view, it was deemed worth while to include in this paper, not only a full discussion of the lower tones, but also a general account of the nodal arrangement associated with the higher tones.

The Equation of Motion.

Let the bar be formed by the revolution of the line $y = \mathrm{A}x$ about the axis of x, the thick extremity of the bar being given by the plane $x = 1$. The displacement at any point x is y, E is the modulus of elasticity and ρ the density of the material. In the case of a thin bar, in which the rotatory inertia can be neglected,* the equation of motion during a lateral vibration is well known to be

---

* E. W. Barnes, 'Phil. Trans.,' A, vol. 206, p. 297 (1906); D. M. Wrinch, 'Phil. Mag.,' p. 161 (1921).

† Presidential Address to the Quekett Microscopical Club, 'Journ. Q.M.C.,' p. 231 (1917); 'Proc. Roy. Soc.,' B, p. 573 (1917).

$$\frac{\partial^2}{\partial x^2}\left(\tfrac{1}{4}\pi \mathrm{E}\mathrm{A}^4 x^4 \frac{\partial^2 y}{\partial x^2}\right)+\pi\rho \mathrm{A}^2 x^2 \frac{\partial^2 y}{\partial t^2} = 0.$$

For a vibration of period $2\pi/p$,

$$\frac{\partial^2 y}{\partial t^2} = -p^2 y,$$

and therefore

$$\frac{\partial^2}{\partial x^2}\left(x^4 \frac{\partial^2 y}{\partial x^2}\right) = \alpha x^2 y,$$

or

$$\frac{\partial^2}{\partial z^2}\left(z^4 \frac{\partial^2 y}{\partial z^2}\right) = z^2 y,$$

if we introduce a new variable z, defined by the equations,

$$z = qx, \qquad q^2 = 4\rho p^2/\mathrm{E}\mathrm{A}^2.$$

This fourth order differential equation can conveniently be exhibited in the form

$$\vartheta(\vartheta-1)z^2\vartheta(\vartheta-1)y = z^4 y,$$

where

$$\vartheta = z\partial/\partial z,$$

or

$$(\vartheta+2)(\vartheta+1)\vartheta(\vartheta-1)y = z^2 y.$$

When it is written in the form

$$[z^{-1}\vartheta(\vartheta+2)]^2 y = y,$$

two solutions of it are seen to be

$$y = \alpha[1-z/1.3+z^2/1.3.2.4-\ldots], \quad \gamma[1+z/1.3+z^2/1.3.2.4+\ldots],$$
$$= \mathrm{A}\mathrm{J}_2(2\surd z)/z, \qquad \mathrm{C}\mathrm{I}_2(2\surd z)/z.$$

The general solution is therefore evidently

$$y = \mathrm{A}\mathrm{J}_2(2\surd z)/z+\mathrm{B}\mathrm{Y}_2(2\surd z)/z+\mathrm{C}\mathrm{I}_2(2\surd z)/z+\mathrm{D}\mathrm{K}_2(2\surd z)/z,$$

where Y and K are the Bessel functions of the second kind.

The constants A, B, C, D, are determined by the conditions of support of the bar. In the case under consideration the end $z = 0$ is free, and the shearing stress and bending moment, which are respectively proportional to

$$z^4\,\partial^2 y/\partial z^2, \qquad \frac{\partial}{\partial z}\left(z^4 \frac{\partial^2 y}{\partial z^2}\right),$$

must vanish. Since the functions Y and K are logarithmically infinite near $z = 0$, B and D must vanish. Further at the thick end, at which $z = ql = \beta$ the conditions for a clamped end require that y and dy/dx should vanish.

---

* Lord Rayleigh, *loc. cit.*, p. 261.

Since

$$\frac{d}{dz} z^{-1} J_2(2\sqrt{z}) = -z^{-3/2} J_3(2\sqrt{z}), \quad \frac{d}{dz} z^{-1} I_2(2\sqrt{z}) = z^{-3/2} I_3(2\sqrt{z}),$$

these conditions can be written

$$AJ_2(2\sqrt{\beta}) + CI_2(2\sqrt{\beta}) = 0,$$

$$AJ_3(2\sqrt{\beta}) - CI_3(2\sqrt{\beta}) = 0,$$

yielding the equation

$$J_2(2\sqrt{\beta})\, I_3(2\sqrt{\beta}) + J_3(2\sqrt{\beta})\, I_2(2\sqrt{\beta}) = 0, \tag{1}$$

as the period equation determining β.

This period equation can easily be thrown into the form used by Kirchhoff,* namely,

$$0 = 1/2\,!\,3\,! - \beta^2/1\,!\,3\,!\,5\,! + \beta^4/2\,!\,4\,!\,7\,! - \beta^6/3\,!\,5\,!\,9\,! + \ldots.$$

We may write the period equation in a slightly more convenient form by employing Bessel functions of the first and second orders, instead of those of the second and third orders. For by the well known recurrence formulæ for these functions,

$$J_1(x) + J_3(x) = 4J_2(x)/x, \qquad I_1(x) - I_3(x) = 4I_2(x)/x.$$

And the period equation becomes, when we substitute for J_3 and I_3,

$$J_2(2\sqrt{\beta})\, I_1(2\sqrt{\beta}) = J_1(2\sqrt{\beta})\, I_2(2\sqrt{\beta}).$$

The first ten roots of this equation were obtained by Airey, who encountered the same equation in the course of an investigation on the vibrations of circular plates. It is a striking fact that this equation occurs in such very different problems. Quoting the first three roots from Airey's† paper,

$$2\sqrt{\beta} = 5{\cdot}906,\ 9{\cdot}197,\ 12{\cdot}402,$$

and returning to the definition of β, we find that the corresponding periods are given by

$$2\pi/p = 16\pi l^2/d \,.\, \sqrt{\frac{\rho}{E}}\,(5{\cdot}906,\ 9{\cdot}197,\ 12{\cdot}402)^{-2}.$$

---

* *Loc. cit.*, p. 826. Kirchhoff gave the first root as $\beta = 8{\cdot}718$. A more accurate value is 8·7202. P. F. Ward, following Kirchhoff, has also investigated this period equation, which occurred in the course of a discussion of the lateral vibrations of a pyramidal bar of square section, vide 'Phil. Mag.,' p. 85 (1913).

† 'Proc. Phys. Soc. London,' p. 225 (1911).

The Nodal Positions.

The displacement of any point on the rod is given by

$$y = \mathrm{AJ}_2(2\sqrt{z}) + \mathrm{CI}_2(2\sqrt{z})$$

where the ratio of A to C has now been implicitly determined in terms of the period associated with the tone under consideration. Thus

$$\mathrm{AJ}_2(2\sqrt{\beta}) + \mathrm{CI}_2(2\sqrt{\beta}) = 0.$$

Hence the displacement is a multiple of

$$\mathrm{J}_2(2\sqrt{z})\,\mathrm{I}_2(2\sqrt{\beta}) - \mathrm{I}_2(2\sqrt{z})\,\mathrm{J}_2(2\sqrt{\beta}).$$

The equation determining the positions of the nodes, at which the displacement is zero, is therefore

$$\mathrm{J}_2(2\sqrt{z})/\mathrm{I}_2(2\sqrt{z}) = \mathrm{J}_2(2\sqrt{\beta})/\mathrm{I}_2(2\sqrt{\beta}). \tag{2}$$

Associated with the first tone, in which $2\sqrt{(\beta)} = 5{\cdot}906$, there is only one root of this equation which is not larger than β, namely, β itself. The clamped end is naturally the only node. In the next tone, in which $2\sqrt{(\beta)} = 9{\cdot}197$, the equation becomes

$$\mathrm{J}_2(2\sqrt{z})/\mathrm{I}_2(2\sqrt{z}) = \mathrm{J}_2(9{\cdot}197)/\mathrm{I}_2(9{\cdot}197) = 0{\cdot}0001752.$$

In order to solve the equation

$$\mathrm{J}_2(2\sqrt{z})/\mathrm{I}_2(2\sqrt{z}) = a, \tag{3}$$

where, as is always the case in these applications, a is very small, since $\mathrm{I}_2(2\sqrt{\beta})$ is large, we have, writing $2\sqrt{z} = x$,

$$\mathrm{J}_2(x) = a\mathrm{I}_2(x)$$

and an approximation to the roots can be obtained by considering the roots of the equation

$$\mathrm{J}_2(x) = 0. \tag{4}$$

For, putting

$$x = \alpha + \epsilon$$

where α is a root of the equation $\mathrm{J}_2(x) = 0$, and neglecting ϵ^2 we get

$$\mathrm{J}_2(\alpha) + \epsilon \mathrm{J}_2'(\alpha) = a\mathrm{I}_2(\alpha) + a\epsilon \mathrm{I}_2'(\alpha);$$

or, since $\mathrm{J}_2(\alpha) = 0$,

$$\begin{aligned}\epsilon &= a\mathrm{I}_2(\alpha)/(\mathrm{J}_2'(\alpha) - a\mathrm{I}_2'(\alpha))\\ &= 2a\mathrm{I}_2(\alpha)/[\mathrm{J}_1(\alpha) - \mathrm{J}_3(\alpha) - a(\mathrm{I}_1(\alpha) + \mathrm{I}_3(\alpha))].\end{aligned}$$

But $\mathrm{J}_1(\alpha) = -\mathrm{J}_3(\alpha)$ since $\mathrm{J}_2(\alpha) = 0$; therefore

$$\epsilon = 2a\mathrm{I}_2(\alpha)/[2\mathrm{J}_1(\alpha) - a(\mathrm{I}_1(\alpha) + \mathrm{I}_3(\alpha))]$$

and evidently this value of ϵ is smaller for the earlier zeros of J_2 than for the later ones. For example, in the case of the second tone, with a node at the clamped end, corresponding to the value $x = 9{\cdot}197$, and another node in the bar, the first root of $J_2(x) = 0$, namely, $x = 5{\cdot}135$, gives a close approximation to the first root of the equation (3).

For putting
$$\alpha = 5{\cdot}135$$
we can determine readily from tables and recurrence formulæ that
$$J_1(\alpha) = -J_3(\alpha) = -0{\cdot}3396,$$
$$I_1(\alpha) = 27, \qquad I_3(\alpha) = 11,$$
and since in this case
$$a = 0{\cdot}000175$$
we find that
$$\epsilon = -0{\cdot}004$$
as a first correction. Thus,
$$x = 5{\cdot}131.$$

The approximation can be continued in the same manner, but it is not in this case necessary. The node is therefore at a distance from the tip, which is in the ratio
$$z/\beta = (5{\cdot}131/9{\cdot}197)^2 = 0{\cdot}304$$
to the length of the bar.

For the third tone, which will have two nodes in the bar, in addition to the node at the clamped end, we have $2\surd\beta = 12{\cdot}402$, and the equation for the nodes, if $x = 2\surd z$, is
$$J_2(x)/I_2(x) = J_2(12{\cdot}402)/I_2(12{\cdot}402) = -0{\cdot}159/23545 = -0{\cdot}00000675.$$
There will be roots of this equation in the neighbourhood of 5·135 and 8·417, the first two zeros of J_2. For the root near 5·135, the necessary correction to the value 5·135 is effectively, for all tones, in the ratio of the corresponding values of a, and thus becomes, for the third tone,
$$\epsilon = aI_2(5{\cdot}135)/J_1(5{\cdot}135) = 0{\cdot}004\,(0{\cdot}0^5675)/0{\cdot}0^3175$$
and is in general negligible. The value of x for the first root of the nodal equation for the third tone is therefore 5·135 effectively. In the next case,
$$a = 0{\cdot}0^6273,$$
and therefore the first root is again effectively 5·135. The node nearest to the tip, therefore, in all tones corresponds to practically the same value of x. The mutual separations of the second roots and third roots of the nodal equations become progressively larger.

The other node in the third tone will be given by a value of x in the neighbourhood of 8·417. For argument so large as 8 the function I_2 is effectively given by the asymptotic expansion
$$I_2(x) = e^x/(2\pi x)^{1/2}$$
and our equation is effectively,
$$x^{1/2}\,J_2(x)\,e^{-x} = a/\surd(2\pi).$$

Although interpolation is not easily accomplished with this equation, on account of the exponential nature of I_2, we find that it gives excellent results in this and similar cases when logarithms are taken. The solution is quickly found to be

$$x = 8{\cdot}401.$$

The corresponding nodal ratio is

$$z/\beta = (8{\cdot}401/12{\cdot}402)^2 = 0{\cdot}458.$$

Thus the nodal ratios in this tone are

$$0{\cdot}171, \quad 0{\cdot}458, \quad 1.$$

A general discussion of the equation

$$J_2(2\sqrt{z})/I_2(2\sqrt{z}) = J_2(2\sqrt{\beta})/I_2(2\sqrt{\beta})$$

is given later, with a more general investigation of the nodal arrangements in the higher tones than can be obtained from these numerical considerations.

The Higher Tones.

Lord Rayleigh,[*] in the course of his investigation into the lateral vibrations of a free-clamped bar of uniform cross-section, quoted a well known formula, due to Seebeck and Donkin, for the positions of the nodes in the higher tones. Seebeck (*loc. cit.*) and Donkin proved that the nodal ratios in the nth tone, measuring from the free end, are given by

$$1{\cdot}3222/(4n-2), \quad 4{\cdot}9820/(4n-2), \quad 9{\cdot}0007/(4n-2), \quad 13/(4n-2),$$
$$17/(4n-2), \ldots \; (4n-10{\cdot}993)/(4n-2),$$
$$(4n-7{\cdot}0175)/(4n-2) \ldots \; (4n-2)/(4n-2).$$

Lord Rayleigh found the corresponding ratios for the free-free bar to be

$$1{\cdot}3222/(4n-2), \quad 4{\cdot}9820/(4n-2), \quad 9{\cdot}0007/(4n-2), \quad 13/(4n-2),$$
$$17/(4n-2), \quad 21/(4n-2), \ldots \; (4n-3)/(4n-2).$$

We proceed to obtain a corresponding formula for the bar of conical type.

The Period Equation.

Referring again to the equation (1), we notice that the period equation is formed by eliminating the constants A and C between the equations

$$y = AJ_2(2\sqrt{z})/z + CI_2(2\sqrt{z})/z = 0, \tag{5}$$

$$\partial y/\partial z = A\partial/\partial z\,(J_2(2\sqrt{z})/z) + C\partial/\partial z\,(I_2(2\sqrt{z})/z) = 0. \tag{6}$$

The period equation is therefore of the form,

$$Y = y_1 y_2' - y_2 y_1' = 0, \tag{7}$$

---

[*] *Loc. cit.*, p. 286.

where $y_1 = J_2(2\sqrt{z})/z = 1-z/1.3+z^2/1.3.2.4-\ldots$

and satisfies the equation

$$\vartheta(\vartheta+2)\,y_1 = -zy_1 \tag{8}$$

where, again, $\vartheta = z\,\partial/\partial z$, and

$$y_2 = I_2(2\sqrt{z})/z = 1+z/1.3+z^2/1.3.2.4+\ldots$$

and satisfies the equation

$$\vartheta(\vartheta+2)\,y_2 = zy_2. \tag{9}$$

We now proceed to find the differential equation satisfied by Y.

We have $y_1\vartheta y_2-y_2\vartheta y_1 = z\mathrm{Y}$.

Applying the operator $(\vartheta+2)$ to both sides, we obtain

$$y_1\vartheta(\vartheta+2)\,y_2-y_2\vartheta(\vartheta+2)\,y_1 = z(\vartheta+3)\,\mathrm{Y}$$

which with the help of equations (8), (9), becomes

$$y_1y_2 = \tfrac{1}{2}(\vartheta+3)\,\mathrm{Y}.$$

Next applying the operator $\vartheta(\vartheta+2)$, we get

$$y_1\vartheta(\vartheta+2)\,y_2+y_2\vartheta(\vartheta+2)\,y_1+2\vartheta y_1\vartheta y_2 = \tfrac{1}{2}\vartheta(\vartheta+2)(\vartheta+3)\,\mathrm{Y},$$

or, again, using the equations (8), (9),

$$\vartheta y_1\vartheta y_2 = \tfrac{1}{4}\vartheta(\vartheta+2)(\vartheta+3)\,\mathrm{Y}.$$

Applying the operator ϑ, the equation yields

$$\vartheta^2 y_1\vartheta y_2+\vartheta y_1\vartheta^2 y_2 = \tfrac{1}{4}\vartheta^2(\vartheta+2)(\vartheta+3)\,\mathrm{Y}.$$

Adding to both sides $4\vartheta y_1\vartheta y_2$ we obtain the equation

$$\vartheta(\vartheta+2)\,y_1\vartheta y_2+\vartheta(\vartheta+2)\,y_2\vartheta y_1 = \tfrac{1}{4}\vartheta^2(\vartheta+2)(\vartheta+3)\,\mathrm{Y}+\vartheta(\vartheta+2)(\vartheta+3)\,\mathrm{Y},$$

or $z(y_2\vartheta y_1-y_1\vartheta y_2) = \tfrac{1}{4}\vartheta(\vartheta+2)(\vartheta+3)(\vartheta+4)\,\mathrm{Y}$,

giving finally the fourth order equation for Y,

$$\vartheta(\vartheta+2)(\vartheta+3)(\vartheta+4)\,\mathrm{Y}+4z^2\mathrm{Y} = 0. \tag{10}$$

The Asymptotic Expansion of the Series for Determining the Period.

The expression of the fourth order equation for Y in terms of the operator ϑ enables us to write down series solutions at once. Since the particular solution we need begins with a constant term, it is clearly

$$\begin{aligned}\mathrm{Y} &= a(1-4z^2/2.4.5.6+4z^{2\cdot 2}/2.4.5.6.4.6.7.8.-\ldots)\\ &= a(1-(\tfrac{1}{2}z)^2/1.2.5/2.3+(\tfrac{1}{2}z)^4/1.2.5/2.3.2.3.7/2.4-\ldots).\end{aligned}$$

This series is of the generalised hypergeometric type with four denominators, namely,

$$1-x/1.1+a.1+b.1+c+x^2/1.2.1+a.2+a.1+b.2+b.1+c.2+c-\ldots$$

and the asymptotic expansion of the series for large values of x is known to be*

$$\frac{\Gamma(1+a)\,\Gamma(1+b)\,\Gamma(1+c)}{2\,(2\pi)^{3/2}}\left\{\cos(4x^{1/4}\sin\pi/4-\pi s_4/4)\,[1+\alpha_2/x^{1/2}+\alpha_4/x+\ldots]\right.$$

$$\left.+\sin(4x^{1/4}\sin\pi/4-\pi s_4/4)\left[\frac{\alpha_1}{x^{1/4}}+\frac{\alpha_3}{x^{3/4}}+\ldots\right]\right\}$$

where $$s_4 = a+b+c+3/2.$$

Taking, in the first instance, only the term of largest order, we may consider the roots of the equation

$$\cos(4x^{1/4}\sin\pi/4-\pi s_4/4) = 0,$$

which may be written

$$4x^{1/4}\sin\pi/4-\pi s_4/4 = (2r-1)\,\pi/2, \qquad (r = 1, 2, \ldots).$$

In the particular series under discussion,

$$x = z^2/4, \qquad s_4 = 6,$$

and the corresponding roots will be

$$2z^{1/2} = (r+1)\,\pi, \qquad (r = 1, 2, \ldots),$$

The first ten roots of the period equation are already known to be

$$x = 5{\cdot}906,\ 9{\cdot}197,\ 12{\cdot}402,\ 15{\cdot}579,\ 18{\cdot}745,$$
$$21{\cdot}901,\ 25{\cdot}055,\ 28{\cdot}205,\ 31{\cdot}354,\ 34{\cdot}502,$$

or, as they may be written more conveniently,

$$x/\pi = 1{\cdot}8807,\ 2{\cdot}9277,\ 3{\cdot}9478,\ 4{\cdot}9591,\ 5{\cdot}9660,$$
$$6{\cdot}9714,\ 7{\cdot}9751,\ 8{\cdot}9781,\ 9{\cdot}9804,\ 10{\cdot}983,$$

whereas the roots obtained by using the first term of the asymptotic expansion are

$$x/\pi = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ \ldots\,.$$

Our approximation, therefore, gives even the fourth root with an error of less than one part in 100. The error in the higher roots rapidly becomes smaller and is less than 1·5 in 1000 in the case of the tenth root.

Consequently, the periods associated with the first ten tones are given by

$$2\pi/p = 16l^2/d\pi\,.\,\sqrt{\frac{\rho}{E}}\,[1{\cdot}8807,\ 2{\cdot}9277,\ 3{\cdot}9478,\ 4{\cdot}9591,\ 5{\cdot}9660,$$
$$6{\cdot}9714,\ 7{\cdot}9751,\ 8{\cdot}9781,\ 9{\cdot}9804,\ 10{\cdot}985]^{-2},$$

and in general the period of the $n-1$th tone, when $n \geqq 11$, is given, with an error of less than three parts in 1000, by

$$2\pi/p = 16l^2/n^2\pi d\,.\,\sqrt{\frac{\rho}{E}}.$$

---

* *Loc. cit.*, Barnes, Wrinch.

The Nodal Positions in the Higher Tones.

The equation determining the nodal ratios in the various tones is

$$\mathrm{J}_2(2\sqrt{z})/\mathrm{I}_2(2\sqrt{z}) = \mathrm{J}_2(2\sqrt{\beta})/\mathrm{I}_2(2\sqrt{\beta}) = a$$

when the value of β associated with the particular tone under consideration is inserted. We may write for convenience

$$x = 2\sqrt{z}, \qquad b = 2\sqrt{\beta},$$

and deal with the equation

$$\mathrm{J}_2(x)/\mathrm{I}_2(x) = \mathrm{J}_2(b)/\mathrm{I}_2(b) = a. \tag{11}$$

The values of b associated with the various tones are

$$b = 5{\cdot}906,\ 9{\cdot}197,\ 12{\cdot}402,\ 15{\cdot}579,\ 18{\cdot}745,$$
$$21{\cdot}901,\ 25{\cdot}955,\ 28{\cdot}205,\ 31{\cdot}154,\ 34{\cdot}502,$$
$$b/\pi = 1{\cdot}881,\ 2{\cdot}928,\ 3{\cdot}948,\ 4{\cdot}959,\ 5{\cdot}966,\ 6{\cdot}971,\ 7{\cdot}975,\ 8{\cdot}978,\ 9{\cdot}980,\ 10{\cdot}983,$$

and the corresponding values of $\mathrm{J}_2(b)$ are alternately negative and positive. Thus in the first, third, ..., $2n-1$th tones, in the equation for the nodal distances, a is negative. In the other tones $\mathrm{J}_2(b)$ and consequently a is positive.

The curve
$$y = \mathrm{J}_2(x)/\mathrm{I}_2(x),$$
is of an oscillatory type and cuts the axis of x in a series of points

$$x = \alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ \alpha_r,\ \alpha_{r+1},\ \ldots$$

which are the roots of the equation, $\mathrm{J}_2(x) = 0$. The first three are given by

$$x = 5{\cdot}135,\ 8{\cdot}417,\ 11{\cdot}620,$$
$$x/\pi = 1{\cdot}597,\ 2{\cdot}679,\ 3{\cdot}699,$$

and the higher ones more and more closely by

$$x/\pi = n+3/4, \qquad (n = 4,\ 5\ \ldots).$$

The amplitude decreases rapidly as x increases. From these characteristics of the curve $y = \mathrm{J}_2(x)/\mathrm{I}_2(x)$ it follows that the values of x which satisfy the equation occur in pairs within the divisions

$$(0\ \alpha_1), \qquad (\alpha_2\ \alpha_3), \quad \ldots \quad (\alpha_{2m}\ \alpha_{2m+1}), \quad \ldots,$$

or within the divisions

$$(\alpha_1\ \alpha_2), \qquad (\alpha_3\ \alpha_4), \quad \ldots \quad (\alpha_{2m+1}\ \alpha_{2m+2}), \quad \ldots,$$

where, as before, the α's are the roots of the equation $\mathrm{J}_2(x) = 0$. Hence, in the $(n-1)$th tone, supposing first that n is odd (so that a is positive,) the roots can be represented by

$$x = \alpha_1-\lambda_1(n), \qquad \alpha_2+\lambda_2(n), \qquad \alpha_3-\lambda_3(n),\ \ldots\ \alpha_n+\lambda_n(n).$$

In this expression for the roots, the quantities λ are all positive. If n is even the roots can be represented in the same way with all the signs of the λ's reversed. It is hardly practicable to find the values of the λ's associated with the higher tones to any specified degree of accuracy. We shall instead deduce general properties of the sequences $\lambda_1, \lambda_2, \ldots, \lambda_r$, associated with each tone.

The fact that the amplitude of the curve

$$y = J_2(x)/I_2(x)$$

decreases in each succeeding division implies that the quantities

$$\lambda_1(r), \lambda_2(r), \ldots, \lambda_r(r)$$

form a steadily increasing sequence. For in order that x may satisfy the equation

$$J_2(x)/I_2(x) = a$$

where a is not zero although it is small, in each division x will have to move further away from the points

$$\alpha_1, \alpha_2, \ldots \alpha_r,$$

which define the divisions and satisfy the equation

$$J_2(\alpha) = 0.$$

The distance between roots belonging to the same division becomes smaller and, at the same time, the distance between roots belonging to different divisions becomes larger. The pairing of the roots, consequently, becomes increasingly marked.

We therefore have the result,

$$0 < \lambda_1(r) < \lambda_2(r) < \lambda_3(r) \ldots < \lambda_r(r).$$

And we observe that as we consider the roots of the equations associated with higher tones, the corresponding λ's increase. Further, the upper limits associated with the first, second, ..., rth tones are given respectively by the differences between the first, second, ..., rth roots of the equations

$$J_2(x)/I_2(x) = a, \qquad J_2(x) = 0,$$

and we therefore have

$$0 < \lambda_1(r) < \lambda_2(r) \ldots < \lambda_r(r) < (b_r - \alpha_r)$$

where b_r and α_r represent the rth roots of these equations. The higher roots of these equations tend to the form $(r+1)\pi$ and $(r+3/4)\pi$ respectively, so that $b_r - \alpha_r$ tends to $\pi/4$.

We collect together the values of x associated with the first three tones, which we have already investigated in an earlier section of the paper.

First tone, 5·906,
Second tone, 5·131, 9·197,
Third tone, 5·135, 8·402, 12·402,

and the first three roots of the equation $J_2(x) = 0$ are

$$5·135, \quad 8·417, \quad 11·620;$$

the corresponding values of the λ's are exhibited in the subjoined table

	λ_1.	λ_2.	λ_3.
First tone	0·771		
Second tone............	0·004	0·780	
Third tone	0·000	0·015	0·782

We may, in addition, exhibit the behaviour of the first and last λ's associated with each tone.

Number of tone.	$\lambda_1(r)$.	$\lambda_r(r)$.
1	0·771	0·771
2	0·004	0·780
3	0·000	0·782
4	0·000	0·783
5	0·000	0·785

The convergence of $\lambda_1(r)$ to zero and the convergence of $\lambda_r(r)$ to $\pi/4$ is plainly seen.

The Nodal Ratios.

Let b_m represent the mth root of the period equation

$$J_2(b)/I_2(b) = J_3(b)/I_3(b),$$

and ${}_mx_n$ the mth root of the nodal equation associated with the nth tone, namely,

$$J_2(x)/I_2(x) = J_2(b_n)/I_2(b_n). \tag{12}$$

We will denote the ratio of the distance of the mth node in the nth tone from the tip to the whole length of the bar by ${}_mr_n$. Then, as before, the nodal ratios will be given by the formula

$${}_mr_n = ({}_mx_n/b_n)^2.$$

Now, we have already shown that in the case of the roots of the nodal equation there exists a sequence

$$\lambda_1(n), \quad \lambda_2(n), \quad \dots \quad \lambda_n(n),$$

which is such that

$$0 < \lambda_1(n) < \lambda_2(n) \dots < \lambda_n(n) < \frac{\pi}{4}$$

which enables us to express the roots ${}_m x_n$ in the form

$${}_m x_n = \alpha_m \pm \lambda_m(n),$$

the upper or lower signs being taken, according as $n-m$ is even or odd. As before, we write α_m for the mth zero of the function $J_2(x)$. If we put

$${}_m R_n = (\alpha_m / b_n)^2$$

it is evident that the nodal ratios interlace with the ratios ${}_m R_n$ in such a way that

$${}_n r_n > {}_n R_n > {}_{n-1} R_n > {}_{n-1} r_n > {}_{n-2} r_n > {}_{n-2} R_n > {}_{n-3} R_n \dots .$$

The nodal ratios, in fact, occur in pairs within the successive pairs

$$({}_{n-1} R_n, \quad {}_{n-2} R_n), \qquad (n_{n-3} R_n, \quad {}_{n-4} R_n) \dots .$$

Further, the ratios ${}_m r_n$ approach the ratios ${}_m R_n$ more and more closely as m decreases. We have already seen that the difference between

$$(\alpha_1 / b_n)^2$$

and the nodal ratio

$$({}_1 x_n / b_n)^2$$

decreases as n increases, and is indeed less than one part in 5000 when $n = 3$. We found also that the ratio

$$(\alpha_2 / b_n)^2$$

differs from the nodal ratio

$$({}_2 x_n / b_n)^2$$

by only one part in 500 for $n = 3$. It is clear, therefore, that when n is sufficiently large a good approximation to the smaller nodal ratios will be given by the ratios

$$(\alpha_1 / b_n)^2, \qquad (\alpha_2 / b_n)^2, \qquad (\alpha_3 / b_n)^2 \dots .$$

In the case of a bar of unit length, the intervals between two nodes in the neighbourhood of the tip is evidently always of order $1/n^2$. For the earlier values of the α's are given by

$$\alpha / \pi = 1{\cdot}597,\ 2{\cdot}679,\ 3{\cdot}699,\ 4{\cdot}710,\ 5{\cdot}717,\ 6{\cdot}721,\ 7{\cdot}725,\ 8{\cdot}730,\ 9{\cdot}731.$$

Further $$b_n = (n+1)\pi + 0(1/n).$$

The nodal ratios in the neighbourhood of the tip are consequently given by

$$(1{\cdot}597/n)^2, \qquad (2{\cdot}679/n)^2, \qquad (3{\cdot}699/n)^2, \dots ,$$

if we neglect terms of order $1/n^3$.

When we pass to nodal ratios in which m is comparable with n, we have

$$\begin{aligned}(\alpha_{n-1}-\pi/4)^2/b_n^2 &< {}_{n-1}r_n < (\alpha_{n-1}/b_n)^2,\\ (\alpha_{n-2}+\pi/4)^2/b_n^2 &> {}_{n-2}r_n > (\alpha_{n-2}/b_n)^2,\\ (\alpha_{n-3}-\pi/4)^2/b_n^2 &< {}_{n-3}r_n < (\alpha_{n-3}/b_n)^2, \dots .\end{aligned}$$

These results specify the nodal ratios within

$$\frac{\pi}{4}(2\alpha_{n-1}-\pi/4)/b_n^2.$$

We can easily deduce that the intervals between consecutive nodes in the neighbourhood of the base are of order $1/n$, in the case of a bar of unit length.

To obtain the values of the nodal ratios to any degree of accuracy which is required, we employ the method of a previous section. As a first approximation to the roots of the equation

$$\mathrm{J}_2(x)/\mathrm{I}_2(x) = \mathrm{J}_2(b_n)/\mathrm{I}_2(b_n) = a_n$$

we obtained the result

$${}_m x_n = \alpha_m + {}_n\epsilon_m,$$

with
$${}_n\epsilon_m = a_n \mathrm{I}_2(\alpha_m)/(\mathrm{J}_1(\alpha_m) - a_n(\mathrm{I}_1(\alpha_m)+\mathrm{I}_3(\alpha_m))).$$

Since

$$\begin{aligned}\alpha_m &= (m+3/4)\pi + 0(1/m),\\ b_n &= (n+1)\pi + 0(1/n),\\ \mathrm{I}_1(\alpha_m) &= \frac{e^{(m+3/4)\pi}}{\sqrt{2\pi^2 m}}\left[1+0\left(\frac{1}{m}\right)\right] = \mathrm{I}_3(\alpha_m),\\ \mathrm{J}_1(\alpha_m) &= \sqrt{\frac{2}{\pi^2 m}}\cos m\pi\left[1+0\left(\frac{1}{m}\right)\right],\end{aligned}$$

and

$$a_n = 2e^{-(n+1)\pi}\cos\left[(n-\tfrac{1}{4})\pi\right]\left[1+0\left(\frac{1}{n}\right)\right],$$

therefore

$${}_n\epsilon_m = (-)^m e^{(m-n-\frac{1}{4})\pi}\left[\cos\left(n\pi-\frac{\pi}{4}\right)+0\left(\frac{1}{m}\right)\right],$$

and

$${}_m r_n = ({}_m x_n/b_n)^2 = \left(\frac{\alpha_m}{b_n}\right)^2\left(1+2\frac{{}_n\epsilon_m}{\alpha_m}\right)+0\left(e^{2(m-n-\frac{1}{4})\pi}/n^2\right).$$

Even when we are considering the node nearest to the clamped base, for which $m = n-1$, the expression

$${}_m r_n = \left(\frac{\alpha_m}{b_n}\right)^2\left(1+2\frac{{}_n\epsilon_m}{\alpha_m}\right)$$

gives the nodal ratio with an error only of order

$$e^{-5\pi/2}/n^2 = 3{\cdot}8820\times 10^{-4}/n^2.$$

This degree of accuracy is adequate to many applications which may arise. A higher degree of accuracy can, however, be obtained by working out a second approximation in the form

$${}_m x_n = \alpha_m + {}_m\epsilon_n + {}_m\eta_n.$$

Other Conditions of Support.

It is worth while to enquire how far the characteristics of the serial arrangement of nodes in the higher tones is dependent upon the particular conditions of support at the base which we have selected for consideration. We propose now to consider the case of a conical rod in which the tip is, as before, free, but in which the conditions at the base are not specified.

The condition that the tip of the rod is free determines that in the equation for the displacement,

$$y = \mathrm{AJ}_2(2\sqrt{z})/z + \mathrm{BY}_2(2\sqrt{z})/z + \mathrm{CI}_2(2\sqrt{z})/z + \mathrm{DK}_2(2\sqrt{z})/z,$$

the constants B and D must vanish, so that the expression for y is of the form

$$y = \mathrm{AJ}_2(2\sqrt{z})/z + \mathrm{CI}_2(2\sqrt{z})/z.$$

The nodal equation will be

$$\mathrm{AJ}_2(2\sqrt{z}) + \mathrm{CI}_2(2\sqrt{z}) = 0,$$

and this equation may be written

$$\mathrm{J}_2(x)/\mathrm{I}_2(x) = -\mathrm{C}/\mathrm{A}, \tag{13}$$

where C/A is some constant depending upon the conditions at the base. Evidently there will be some characteristic constant—which we may write $a(n)$—associated with each tone. From our discussion of the nodal equation in the case when the base of the rod is clamped, namely,

$$\mathrm{J}_2(x)/\mathrm{I}_2(x) = \mathrm{J}_2(b_n)/\mathrm{I}_2(b_n),$$

it is evident that the roots of the nodal equation (13) will be arranged in pairs within successive pairs of the roots of the equation

$$\mathrm{J}_2(x) = 0.$$

Further, if

$$\mathrm{J}_2(x)/\mathrm{I}_2(x) = a(n)$$

be the nodal equation for the nth tone, since it must have n roots, it is evident that we must have

$$a(n) = \mathrm{J}_2({}_n x_n)/\mathrm{I}_2({}_n x_n)$$

where ${}_n x_n$ lies between $(n-\frac{1}{4})\pi$ and $(n+\frac{1}{4})\pi$. We can, therefore, estimate the order of magnitude of $a(n)$, and as before we find that we get the nodes in the neighbourhood of the tip to a high degree of accuracy by putting

$${}_m x_n = \alpha_m.$$

It is obvious that the nth root of the period equation, which we may denote by c_n, must be such as to make

$${}_nx_n/c_n \leq 1, \qquad ({}_nx_n+2\pi)/c_n > 1,$$

for otherwise there would be $(n+1)$ nodal ratios associated with the n tone. But as the nth root of an equation of the type

$$J_2(x)/I_2(x) = a,$$

lies whatever a may be, between $(n-\frac{1}{4})\,\pi$ and $(n+\frac{1}{4})\,\pi$, therefore

$$(n+3\tfrac{3}{4})\,\pi > c_n \geq (n-\tfrac{1}{4})\,\pi.$$

Now, b_n, the nth root of the period equation in the case when the base of the rod is clamped, was found to be given by

$$b_n = (n+1)\,\pi + 0\left(\frac{1}{n}\right).$$

Therefore $$c_n = b_n + 0\left(\frac{1}{n}\right) + \epsilon$$

where $$-5\pi/4 < \epsilon < 11\pi/4.$$

Hence $$({}_mx_n/c_n)^2 = ({}_mx_n/b_n)^2\,(b_n/c_n)^2 = ({}_mx_n/b_n)^2\left(1+0\left(\frac{1}{b_n}\right)\right).$$

The nodal ratios in the neighbourhood of the tip are then evidently as before,

$$(\alpha_1/b_n)^2, \qquad (\alpha_2/b_n)^2, \qquad (\alpha_3/b_n)^2 \ldots$$

if we neglect terms of order $1/n^3$. The intervals between consecutive nodes, in this neighbourhood, are clearly of order $1/n^2$.

Although we cannot specify the nodal positions in the other parts of the bar without some data as to the conditions of support at the base, we can see that the intervals between consecutive nodes, in the neighbourhood of the base, are of order $1/n$. Thus, as in the special case of the bar with a clamped base, the nodes crowd together with intervals of order $1/n^2$ near the tip, and becomes progressively further apart, the intervals being of order $1/n$ in the neighbourhood of the base.